Supplementary Materials

"Health Risk and the Value of Life"

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This document uses a three-state version of the continuous model introduced in Section 2 of the main text to provide additional insight regarding value function concavity. We solve the model analytically by assuming that utility takes the isoelastic form employed in Section 3 of the main paper. The results for the three-state example are available in an accompanying spreadsheet (`healthrisk.supplement.xlsx`) that calculates the value of statistical life (VSL), VSL per quality-adjusted life-year (QALY), the value of statistical illness (VSI), and VSI per QALY for a particular set of parameters. The two main components of the spreadsheet are:

1. **The yellow highlighted cells**: adjustable parameters that can be specified by the user

2. **The orange highlighted cells**: positive values indicate that the value function concavity condition (11) from the main text is satisfied

Proposition 7 from the main text states the VSI per QALY is increasing in disease severity if and only if the value function is concave in health states. Value function concavity, given by equation (11) in the main text, will typically be satisfied when health differences across states are large. The spreadsheet can be used to confirm this theoretical intuition and to investigate what drives violations of value function concavity.

The last section of this document provides a set of sufficient conditions for value function concavity to hold. It shows that if (i) utility takes the isoelastic form employed in Section 3 of the main paper; (ii) mortality rates are constant within states and increase in worse health states; and (iii) quality of life is non-decreasing in health states states, then value function concavity will always hold provided that transition rates between states are sufficiently small. We show that the proof can be generalized to a model with an arbitrary number of health states.

Model

The example assumes that there are $n = 3$ health states, with absorbing state $n+1 = 4$ denoting death. Health states 1, 2, and 3 can be interpreted as "healthy," "mildly-ill," and "severely ill" under our default parameter choices. The spreadsheet allows you to specify the mortality rates for each of these three states (λ_{14}, λ_{24}, and λ_{34}). You can also specify the transition rate from state 1 to 2 and from state 1 to 3. For simplicity, we assume those two transition rates are equal ($\lambda = \lambda_{12} = \lambda_{13}$), and that the transition rate from state 2 to 3 (λ_{23}) is equal to 0. We further assume that all transition rates are time-invariant ($\lambda_{ij}(t) = \lambda_{ij}$), and that the interest and discount rates are zero ($r = \rho = 0$). Initial wealth (W_0) can be specified in the spreadsheet.

The spreadsheet uses the utility specification from Section 3 of the main paper:

$$u(c,q) = q\left(\frac{c^{1-\gamma} - \underline{c}^{1-\gamma}}{1-\gamma}\right)$$

The spreadsheet reports results for the utility parameters $\gamma = 2$ and $\gamma = 1/2$. The subsistence level of consumption ($\underline{c}$) can specified in the spreadsheet. Quality of life for each of the three health states can also be specified in the spreadsheet. We assume that quality of life is time invariant ($q_i(t) = q_i$).

Examples

Below, we discuss a few results, which can be verified by changing the parameters in the spreadsheet:

- Under the default parameters, we have that $\frac{VSI(i,j)}{D_i - D_j} < \frac{VSI(i,k)}{D_i - D_k}$ where $D_i > D_j > D_k$ for both utility specifications $\gamma = 2$ and $\gamma = 1/2$. From the first of part of Proposition 7, this inequality implies that value function concavity is satisfied, as is confirmed by the positive values in cells D21 and D30 of the spreadsheet. Under the default parameters, quality of life is constant, and there is a strong ordering in mortality rates across states. Note that state dependence has no effect on value function concavity when quality of life is constant across states. Thus, this result is consistent with theoretical intuition.

- Value function concavity no longer holds if mortality in state 1 becomes closer to mortality in state 2 (e.g., set $\lambda_{14} = 0.09$). This violation occurs because $\lambda_{13} > 0$ but $\lambda_{23} = 0$: it is no longer clear whether state 2 is "worse" than state 1, because while state 2's mortality is slightly higher than state 1, the probability of transitioning to the worst state ($i = 3$) is higher in state 1. It is easy to generate other examples with violations of concavity, including situations where value function concavity is satisfied for one risk aversion level but not the other.

- When reducing quality of life in states 2 and 3 (e.g., setting $q_2 = q_3 = 0.6$), we also obtain a violation in value function concavity. This violation occurs because utility exhibits positive state dependence. However, the results are subtle. For instance, decreasing quality of life in state 3 from 0.6 to 0.1 yields value function concavity.

Users can use the spreadsheet to investigate additional scenarios as they wish.

Derivations

This section derives the formulas that are used in the spreadsheet. We commence by solving optimal consumption levels across states. Here, we ignore the subsistence level of consumption, $\underline{c}$, as optimal consumption paths will not depend on it. For states $i = 2, 3$, the HJB equation (2) from the main text takes the form:

$$\lambda_{i4}\, V(w,i) = \max_{c_i} \left\{ q_i\, c_i^{1-\gamma}/(1-\gamma) - V_w(w,i)\, c_i \right\}$$

The value function, V, and the level of consumption, c, are time-invariant because the transition rates are constant. Assuming $V(w,i) = K_i\, w^{1-\gamma}/(1-\gamma)$, we obtain $c(w,i) = (K_i/q_i)^{-1/\gamma}\, w$. Plugging that back in and solving for K_i yields:

$$c(w,i) = \lambda_{i4}/\gamma \text{ and } V(w,i) = \underbrace{q_i\, (\lambda_{i4}/\gamma)^{-\gamma}}_{K_i} w^{1-\gamma}/(1-\gamma)$$

Moving on to state $i = 1$ and exploiting the forms of the value functions in states 2 and 3, the HJB equation (2) takes the form:

$$(\lambda_{14} + 2\lambda)\, V(w,i) = \max_{c_1} \left\{ q_1\, \frac{c_1^{1-\gamma}}{1-\gamma} - V_w(w,1)\, c_1 \right\} + \lambda\, \left(q_2\, (\lambda_{24}/\gamma)^{-\gamma} + q_3\, (\lambda_{34}/\gamma)^{-\gamma}\right) \frac{w^{1-\gamma}}{1-\gamma}$$

where $\lambda = \lambda_{12} = \lambda_{13}$. Again assuming $V(w,i) = K_i\, w^{1-\gamma}/(1-\gamma)$, $c_1 = [V(w,1)/q_1]^{-1/\gamma}$, we obtain:

$$0 = \gamma\, q_1\, [K_1/q_1]^{1-1/\gamma} - [\lambda_{14} + 2\lambda]\, K_1 + \lambda\, [q_2\, (\lambda_{24}/\gamma)^{-\gamma} + q_3\, (\lambda_{34}/\gamma)^{-\gamma}]$$

Applying the quadratic formula, we can solve for K_1 when setting $\gamma = 2$:

$$K_1 = \left(\frac{\sqrt{q_1} + \sqrt{q_1 + [\lambda_{14} + 2\lambda]\lambda[q_2\,(2/\lambda_{24})^2 + q_3\,(2/\lambda_{34})^2]}}{\lambda_{14} + 2\lambda} \right)^2$$

and when setting $\gamma = 1/2$:

$$K_1 = \frac{\lambda \left[q_2 \sqrt{\frac{1}{2\lambda_{24}}} + q_3 \sqrt{\frac{1}{2\lambda_{34}}} \right] + \sqrt{\lambda^2 \left[q_2 \sqrt{1/\lambda_{24}} + q_3 \sqrt{1/\lambda_{34}} \right]^2 + 2q_1^2 [\lambda_{14} + 2\lambda]}}{2(\lambda_{14} + 2\lambda)}$$

Quality-adjusted life expectancies for states $i = 2, 3$ in this setting are:

$$D_i = \int_0^\infty e^{-\lambda_{i4} t}\, q_i \, dt = q_i / \lambda_{i4}$$

Quality-adjusted life expectancy for state $i = 1$ is:

$$D_1 = \int_0^\infty e^{-(\lambda_{14} + 2\lambda) t} (q_1 + \lambda D_2 + \lambda D_3)\, dt = \frac{q_1}{\lambda_{14} + 2\lambda} + \frac{\lambda q_2}{\lambda_{24}(\lambda_{14} + 2\lambda)} + \frac{\lambda q_3}{\lambda_{34}(\lambda_{14} + 2\lambda)}$$

Using the optimal consumption levels above and inserting the subsistence level into the utility function yields the final value function:

$$V(W_0, i) = K_i \frac{W_0^{1-\gamma}}{1-\gamma} - D_i \frac{\underline{c}^{1-\gamma}}{1-\gamma}$$

We can use this functional form in conjunction with the results for optimal consumption levels above with the formulas for VSL (7) and VSI (8) from the main text. Furthermore, using states $i = 1$, $j = 2$, $k = 3$, the value function concavity condition (11) from the main text takes the form:

$$\frac{1}{1-\gamma} \left(K_2 - \frac{D_2 - D_3}{D_1 - D_3} K_1 - \frac{D_1 - D_2}{D_1 - D_3} K_3 \right) > 0. \tag{1}$$

Sufficient Conditions for Value Function Concavity

Using the above relationships, we can derive sufficient conditions for value function concavity in this setting.

Note that for $\lambda = 0$, we have:

$$D_i = q_i / \lambda_{i4}, \; K_i = q_i \, (\lambda_{i4}/\gamma)^{-\gamma}, \; i = 1, 2, 3$$

Plugging this into the value concavity condition (1) above and simplifying yields:

$$
\begin{aligned}
\frac{1}{(1-\gamma)(D_1-D_3)}\left(\frac{q_1 q_2}{\lambda_{24}^{\gamma}\lambda_{14}}-\frac{q_2 q_3}{\lambda_{24}^{\gamma}\lambda_{34}}-\frac{q_1 q_2}{\lambda_{14}^{\gamma}\lambda_{24}}+\frac{q_1 q_3}{\lambda_{14}^{\gamma}\lambda_{34}}-\frac{q_1 q_3}{\lambda_{34}^{\gamma}\lambda_{14}}+\frac{q_2 q_3}{\lambda_{34}^{\gamma}\lambda_{24}}\right) &> 0 \\
\Leftrightarrow \frac{1}{(1-\gamma)(D_1-D_3)}\Big(q_1 q_2 \lambda_{34}^{\gamma}\lambda_{14}^{\gamma-1}-q_2 q_3 \lambda_{14}^{\gamma}\lambda_{34}^{\gamma-1}-q_1 q_2 \lambda_{34}^{\gamma}\lambda_{24}^{\gamma-1} & \\
+q_1 q_3 \lambda_{24}^{\gamma}\lambda_{34}^{\gamma-1}-q_1 q_3 \lambda_{24}^{\gamma}\lambda_{14}^{\gamma-1}+q_2 q_3 \lambda_{14}^{\gamma}\lambda_{24}^{\gamma-1}\Big) &> 0
\end{aligned}
$$

This equation will be satisfied if λ_{i4}, $i=1,2,3$ is increasing in i (i.e., mortality rates are higher in sicker states) and weakly increasing in q_i, $i=1,2,3$ (i.e., quality of life is constant or improves in sicker states).[1]

To illustrate, consider the case of constant quality of life ($q_i = q$) and $\gamma = 2$. Then the equation above becomes:

$$
\begin{aligned}
\lambda_{34}^2\lambda_{14}-\lambda_{14}^2\lambda_{34}-\lambda_{34}^2\lambda_{24}+\lambda_{24}^2\lambda_{34}-\lambda_{24}^2\lambda_{14}+\lambda_{14}^2\lambda_{24} &< 0 \\
\Leftrightarrow (\lambda_{14}+k+l)^2\lambda_{14}-\lambda_{14}^2(\lambda_{14}+k+l)-(\lambda_{14}+k+l)^2(\lambda_{14}+k) & \\
+(\lambda_{14}+k)^2(\lambda_{14}+k+l)-(\lambda_{14}+k)^2\lambda_{14}+\lambda_{14}^2(\lambda_{14}+k) &< 0 \\
\Leftrightarrow -\lambda_{14}^2(\lambda_{14}+k+l)-(\lambda_{14}+k+l)^2 k+(\lambda_{14}+k)^2(k+l)+\lambda_{14}^2(\lambda_{14}+k) &< 0 \\
\Leftrightarrow -k^2 l - l^2 k &< 0
\end{aligned}
$$

where $\lambda_{24}-\lambda_{14}=k$ and $\lambda_{34}-\lambda_{24}=l$. This inequality is clearly satisfied due to our assumption that mortality rates are constant and increasing in i.

Now note that for $\lambda \geq 0$ and fixed λ_{i4} and q_i, $i=1,2,3$, D_1 and K_1 are continuous functions of λ, $D_1(\lambda)$, and $K_1(\lambda)$. Hence, since

$$
\frac{1}{1-\gamma}\left(K_2-\frac{D_2-D_3}{D_1(0)-D_3}K_1(0)-\frac{D_1(0)-D_2}{D_1(0)-D_3}K_3\right)>0,
$$

there exists a λ_0 such that

$$
\frac{1}{1-\gamma}\left(K_2-\frac{D_2-D_3}{D_1(\lambda)-D_3}K_1(\lambda)-\frac{D_1(\lambda)-D_2}{D_1(\lambda)-D_3}K_3\right)>0
$$

for all $\lambda \in [0,\lambda_0]$.

The same logic applies for arbitrary states $i<j<k<n+1$ in a multi-state setting. Un-

[1] As in Murphy and Topel (2006), the utility function we employ exhibits negative state dependence: the marginal utility of consumption rises with quality of life, q.

der the assumption of a zero transition rate between the states, increasing hazard rates $\lambda_{\cdot n+1}$ and increasing quality $q_{\cdot}$ will imply value function concavity. Then, just as above, one can obtain value function concavity for transitions up to a given threshold by continuity.